Partial Differential Equations Lecture Notes (2024/2025)

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Contents

1 Introduction

Notation

$$u_t := \frac{\partial u}{\partial t} \qquad u_{xx} := \frac{\partial^2 u}{\partial x^2} \qquad u_{xy} = \frac{\partial}{\partial x}\frac{\partial}{\partial y}u$$

Definition *Classical solution*

A **classical solution** of a PDE in n variables is a function $u : D \subset \mathbb{R}^n \to \mathbb{R}$ which:

- satisfies the PDE at every point of D
- is sufficiently smooth (continuously differentiable up to the order of the PDE)

Definition *Linear differential operator*

A **linear differential operator** on $\mathbb{R}^n$ of order m is an expression of the form

$$L[u] = \sum_{k_1+\cdots+k_n<m} a_{k_1,\ldots,k_n}(x_1,\ldots,x_n)\frac{\partial^{k_1+\ldots+k_n}u}{\partial x_1^{k_1}\cdots\partial x_n^{k_n}}$$

Definition *Homogeneous linear PDE*

A **homogeneous linear PDE** is of the form $L[u] = 0$ where L is a linear differential operator.

Proposition *Superposition principle*

If $u_1,\ldots,u_k$ are solutions, then so is

$$u = c_1u_1 + \cdots + c_ku_k$$

where $c_1,\ldots,c_k$ are constant.

Definition *Inhomogeneous linear PDE*

An **inhomogeneous linear PDE** is of the form $L[u] = f$ where L is a linear diff. operator and f a given function.

Theorem

If u_p is a solution to $L[u] = f$, then all solutions of $L[u] = f$ are of the form $u = u_n + u_p$ where $L[u_h] = 0$.

2 Linear and nonlinear waves

2.1 Transport equations

2.1.1 Uniform transport

Proposition *Stationary transport*

Let $D \subset \mathbb{R}^2$ and $D_a := D \cap (\mathbb{R} \times \{a\})$.
If D_a is empty or connected for all $a \in \mathbb{R}$, and u is a classical solution of $\frac{\partial u}{\partial t}$ on D, then u only depends on x.

Proposition *Uniform transport*

If $u : \mathbb{R}^2 \to \mathbb{R}$ is a classical solution of

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

and is defined on all of $\mathbb{R}^2$, then

$$u(t,x) = v(x - ct)$$

where v is a C^1 function of the **characteristic variable** $\xi = x - ct$

Theorem

For a C^1 function $f:\mathbb{R}\to\mathbb{R}$ the **initial value problem**

$$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=0 \qquad u(0,x)=f(x)$$

has solution $u(t,x)=f(x-ct)$.
This solution u is a travelling wave with velocity c, and is constant along **characteristic lines** $x=ct+k$, $k\in\mathbb{R}$

Corollary

The initial value problem

$$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}+au=0 \qquad u(0,x)=f(x)$$

has solution $u(t,x)=f(x-ct)e^{-at}$. Note: u is not constant along characteristic lines.

2.1.2 Nonuniform transport

Definition *Characteristic curve*

Assume $c:\mathbb{R}\to\mathbb{R}$ is continuous and consider

$$\frac{\partial u}{\partial t}+c(x)\frac{\partial u}{\partial x}=0 \qquad (*)$$

The graph of a solution of $\dfrac{\partial x}{\partial t}=c(x)$ is called a **characteristic curve** for $(*)$.

Properties of the equation $\frac{\partial x}{\partial t}=c(x)$

- Horizontal translations of solution curves are again solution curves
- Nonconstant solutions are strictly monotone functions of t
- Nonconstant solutions can be expressed as both $x(t)$ and $t(x)$.

Proposition *Classification of characteristic curves*

Nonconstant solutions of $\frac{\partial x}{\partial t}=c(x)$ are of the form:

$$\beta(x):=\int\frac{1}{c(x)}\,\mathrm{d}x=t+k \qquad k\in\mathbb{R}$$

Characteristic curve: $t\mapsto(t,x(t))=(t,\beta^{-1}(t+k))$ **Characteristic variable**: $\xi=\beta(x)-t$

These solutions can be computed using separation of variables:

$$\frac{\mathrm{d}x}{\mathrm{d}t}=c(x)\implies\int\frac{1}{c(x)}\,\mathrm{d}x=\int 1\,\mathrm{d}t$$

Note: if $c(\overline{x})=0$, then $t\mapsto(t,\overline{x})$ is also a characteristic curve.

Proposition

A solution $u:\mathbb{R}^2\to\mathbb{R}$ of $(*)$ is constant along characteristic curves.

Corollary

A solution $u:\mathbb{R}^2\to\mathbb{R}$ of $(*)$, is of the form

$$u(t,x)=v(\xi)=v(\beta(x)-t)$$

for some C^1 function $v:\mathbb{R}\to\mathbb{R}$.
The initial condition $u(0,x)=f(x)$ gives $u(t,x)=f(\beta^{-1}(\xi))=f(\beta^{-1}(\beta(x)-t))$.

Corollary

If a characteristic curve passes through (t,x) and $(0,y)$, then $u(t,x)=f(y)$.

2.2 The wave equation

Definition *Wave operator*

The **wave operator** is the differential operator given by

$$\Box = \frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} \qquad c > 0$$

The 1-dimensional **wave equation** is given by

$$\Box u = 0 \iff \frac{\partial u^2}{\partial t^2} = c^2 \frac{\partial u^2}{\partial x^2}$$

2.2.1 d'Alembert's formula

Lemma

If u is C^2, then

$$\Box u = (\partial_t - c\partial_x)(\partial_t + c\partial_x)u$$

Theorem *Solutions of the wave equation*

Every classical solution of $\Box u = 0$ can we written as

$$u(t, x) = p(x - ct) + q(x + ct)$$

where p and q are C^2 functions.

Theorem *d'Alembert's formula*

The solution of the initial value problem

$$\Box u = 0 \qquad u(0, x) = f(x) \qquad \frac{\partial u}{\partial t}(0, x) = g(x)$$

is given by

$$u(t, x) = \frac{f(x - ct) + f(x + ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(z)\,\mathrm{d}z$$

2.2.2 External forcing

Proposition

The solution of the initial value problem:

$$\Box u = F(t, x) \qquad u(0, x) = 0 \qquad \frac{\partial u}{\partial t}(0, x) = 0$$

is given by

$$u(t, x) = \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} F(s, y)\,\mathrm{d}y\,\mathrm{d}s$$

Corollary

The solution of the initial value problem:

$$\Box u = F(t, x) \qquad u(0, x) = f(x) \qquad \frac{\partial u}{\partial t}(0, x) = g(x)$$

is given by

$$u(t, x) = \frac{f(x - ct) + f(x + ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(z)\,\mathrm{d}z + \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} F(s, y)\,\mathrm{d}y\,\mathrm{d}s$$

3 Fourier series

3.1 Evolution equations

Definition *Linear evolution equation*

A **linear evolution equation** is of the form

$$\frac{\partial u}{\partial t} = L[u]$$

where the operator L satisfies

$$L[u+v] = L[u] + L[v] \qquad L[c(t)u] = c(t)L[u]$$

Eigenfunctions and eigenvalues

The educated guess $u(t,x) = e^{\lambda t}v(x)$ gives

$$\frac{\partial u}{\partial t} = L[u] \iff \lambda v = L[v]$$

v is called an **eigenfunction** corresponding to the **eigenvalue** λ.

3.2 Fourier series

Definition *Inner product*

Let X be a linear space over $\mathbb{K}$. A map $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{K}$ is called an **inner product** if:

1. $\langle x, x \rangle \geq 0$
2. $\langle x, x \rangle = 0 \iff x = 0$
3. $\langle \lambda x + \mu y, z \rangle = \lambda \langle x, z \rangle + \mu \langle y, z \rangle$ for all $\lambda, \mu \in \mathbb{K}$
4. $\langle x, y \rangle = \overline{\langle y, x \rangle}$

Definition *Orthonormal basis*

Let X be a Hilbert space. The set $\{e_k : k \in \mathbb{N}\}$ is called an **orthonormal basis** for X if

$$\overline{\operatorname{span}\{e_k : k \in \mathbb{N}\}} = X \qquad \langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Definition $L^2(-\pi, \pi)$

$L^2(-\pi, \pi)$ is the completion of the inner product space

$$\{f : [-\pi, \pi] \to \mathcal{C} : f \text{ is continuous }\} \qquad \langle f, g \rangle = \int_{-\pi}^{\pi} f(x)\overline{g(x)}\,\mathrm{d}x$$

Proposition

The functions $\{1, \sin(x), \cos(x), \sin(2x), \cos(2x), \ldots\}$ for $k \in \mathbb{N}$ form an orthogonal basis for L^2.
The same is true for the functions $\{e^{ikx} : k \in \mathbb{Z}\}$.

Theorem *Fourier series*

Any $f \in L^2(-\pi, \pi)$ can be written as a **Fourier series**:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos(kx) + b_k \sin(kx)) \qquad a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,\mathrm{d}x \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,\mathrm{d}x$$

The Fourier series converges with respect to the L^2 norm: $\lim_{n\to\infty} \left(\int_{-\pi}^{\pi} |f(x) - s_n(x)|^2\,\mathrm{d}x \right)^{1/2} = 0$

Corollary *Complex Fourier series*

The Fourier series can also be written as:

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} \qquad c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}\,\mathrm{d}x$$

Lemma

$a_k \to 0$ and $b_k \to 0$ as $k \to \infty$.

Lemma *Periodic extensions*

Let $f : (-\pi, \pi] \to \mathbb{C}$ be any function. Then there exists a function $\tilde{f} : \mathbb{R} \to \mathbb{C}$ such that

- $\tilde{f}|_{(-\pi,\pi]} = f$
- $\tilde{f}$ is 2π-periodic

3.3 Convergence

3.3.1 Pointwise convergence

Definition *Left-hand and right-hand limits*

For $f : \mathbb{R} \to \mathbb{R}$ we say that $\lim_{x\to a^-} f(x) = L$, denoted $L = f(a^-)$, if:

$$\text{for all } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that} \qquad a - \delta < x < a \implies |f(x) - L| < \varepsilon$$

We say that $\lim_{x\to a^+} f(x) = R$, denoted $R = f(a^+)$, if:

$$\text{for all } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that} \qquad a < x < a + \delta \implies |f(x) - R| < \varepsilon$$

Definition *Piecewise continuity*

$f : [a, b] \to \mathbb{R}$ is **piecewise continuous** if it is defined and continuous except at finitely many points

$$a \le x_1 < x_2 < \cdots < x_n \le b$$

and at each x_k the left-hand and right-hand limits of f exist.

A function $f : \mathbb{R} \to \mathbb{R}$ is piecewise continuous if it is piecewise continuous on any compact interval.

Definition *Piecewise smoothness*

$f : [a, b]$ is **piecewise** C^1 if it is defined, continuous and continuously differentiable except at finitely many points

$$a \le x_1 < x_2 < \cdots < x_n \le b$$

and at each x_k the left-hand and right-hand limits of f and f' exist.

A function $f : \mathbb{R} \to \mathbb{R}$ is piecewise C^1 if it is piecewise C^1 on any compact interval.

Theorem *Pointwise convergence of Fourier series*

Assume $f : \mathbb{R} \to \mathbb{R}$ is 2π-periodic and piecewise C^1. Let s_n denote the partial sums of the Fourier series.
Then for all fixed $x \in \mathbb{R}$ we have

$$\lim_{n\to\infty} s_n(x) = \frac{f(x^+) + f(x^-)}{2}$$

If f is continuous at x, then $\lim_{n\to\infty} s_n(x) = f(x)$.

3.3.2 Uniform convergence

Theorem *Uniform convergence of Fourier series*

Let $f : [-\pi, \pi] \to \mathbb{R}$ have Fourier coefficients a_k and b_k, and

$$\sum_{k=1}^{\infty} (|a_k| + |b_k|) < \infty$$

Then the Fourier series of f converges uniformly on $\mathbb{R}$.
The limit $\tilde{f} : \mathbb{R} \to \mathbb{R}$ is continuous and 2π-periodic and has the same Fourier coefficients as f.

Proposition *Convergence rate*

If $f : \mathbb{R} \to \mathbb{R}$ is 2π-periodic and C^n, then there exists M such that $|f^{(n)}(x)| \leq M$ for all $x \in \mathbb{R}$, and

$$|a_k| \leq \frac{2M}{k^n} \qquad |b_k| \leq \frac{2M}{k^n} \qquad \text{for all } k \in \mathbb{N}$$

Theorem

Let $f : [-\pi, \pi] \to \mathbb{R}$ have Fourier coefficients a_k and b_k, and

$$\sum_{k=1}^{\infty} k^n (|a_k| + |b_k|) < \infty$$

Then the limit $\tilde{f} : \mathbb{R} \to \mathbb{R}$ is a C^n function.

3.4 Change of scale

Proposition *Change of scale (real)*

Any $f \in L^2(-\ell, \ell)$ can be written as

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos\left(\frac{k\pi x}{\ell}\right) + b_k \sin\left(\frac{k\pi x}{\ell}\right) \right]$$

$$a_k = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos\left(\frac{k\pi x}{\ell}\right) \mathrm{d}x \qquad b_k = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin\left(\frac{k\pi x}{\ell}\right) \mathrm{d}x$$

Proposition *Change of scale (complex)*

Any $f \in L^2(-\ell, \ell)$ can be written as

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{k\pi i x/\ell} \qquad c_k = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x) e^{-k\pi i x/\ell} \mathrm{d}x$$

Proposition *Fourier series of even and odd functions*

If $f(x)$ is even, then $b_k = 0$, and so $f(x)$ can be represented by a **Fourier cosine series**

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \qquad a_k = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(kx) \mathrm{d}x$$

If $f(x)$ is odd, then $a_k = 0$, and so $f(x)$ can be represented by a **Fourier sine series**

$$f(x) \sim \sum_{k=1}^{\infty} b_k \sin kx \qquad b_k = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(kx) \mathrm{d}x$$

4 Separation of variables

4.1 The heat equation

Derivation of the heat equation

We have the following physical quantities and material properties:

- $\varepsilon(t,x)$ = thermal energy densitiy
- $w(t,x)$ = heat flux
- $\rho(x)$ = mass density
- $\chi(x)$ = specific heat
- $\kappa(x)$ = thermal conductivity

Conservation law:

$$\frac{\partial \varepsilon}{\partial t} + \frac{\partial w}{\partial x} = 0 \implies \frac{\mathrm{d}}{\mathrm{d}t}\int_a^b \varepsilon(t,x)\,\mathrm{d}x = w(t,a) - w(t,b)$$

Constitutive assumption:

$$\varepsilon(t,x) = \rho(x)\chi(x)u(t,x)$$

Fourier's law of cooling:

$$w(t,x) = -\kappa(x)\frac{\partial u}{\partial x}$$

Definition *Heat equation*

1-dimensional heat equation:

$$\frac{\partial}{\partial t}[\rho(x)\chi(x)u(t,x)] + \frac{\partial}{\partial x}\left[-\kappa(x)\frac{\partial u}{\partial x}\right]$$

If ρ, χ, κ are constant, then

$$\frac{\partial u}{\partial t} = \gamma\frac{\partial^2 u}{\partial x^2} \qquad \gamma = \frac{\kappa}{\rho\chi}$$

We have the following initial condition:

$$u(0,x) = f(x)$$

and we choose one or multiple of the following **homogeneous boundary conditions**:

Dirichlet condition: $u(t,a) = 0$ **Neumann condition**: $\frac{\partial u}{\partial x}(t,a) = 0$

Robin condition: $\frac{\partial u}{\partial x}(t,a) + \beta u(t,a) = 0$

We can also impose **periodic boundary conditions**:

$$u(t,a) = u(t,b) \qquad \frac{\partial u}{\partial x}(t,a) = \frac{\partial u}{\partial x}(t,b)$$

Proposition *Homogenisation trick (Dirichlet condition)*

Assume u satisfies the heat equation and

$$u(0,x) = f(x) \qquad u(t,a) = u_a \qquad u(t,b) = u_b$$

Define the function

$$u^*(t,x) = u_a + \frac{u_b - u_a}{b-a}(x-a)$$

Then $v - u - u^*$ satisfies the heat equation and

$$v(0,x) = f(x) - u^*(0,x) \qquad v(t,a) = 0 \qquad v(t,b) = 0$$

Proposition *Homogenisation trick (Neumann condition)*

Assume u satisfies the heat equation and

$$u(0,x) = f(x) \qquad \frac{\partial u}{\partial x}(t,x) = \phi_a \qquad \frac{\partial u}{\partial x}(t,b) = \phi_b$$

Define the function

$$u^*(t,x) = \frac{\phi_b(x-a)^2 - \phi_a(x-b^2) + 2(\phi_b - \phi_a)\gamma t}{2(b-a)}$$

Then $v - u - u^*$ satisfies the heat equation and

$$v(0,x) = f(x) - u^*(0,x) \qquad \frac{\partial v}{\partial x}(t,a) = 0 \qquad \frac{\partial v}{\partial x}(t,b) = 0$$

Definition *Hyperbolic functions*

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\sinh x = \cosh x \qquad \frac{\mathrm{d}}{\mathrm{d}x}\cosh x = \sinh x$$

Solution method

For $a \leq x \leq b$ and $t \geq 0$, consider

$$\frac{\partial u}{\partial t} = \gamma \frac{\partial^2 u}{\partial x^2} \qquad u(0,x) = f(x) \qquad + \text{ homogeneous boundary conditions}$$

The educated guess $u(t,x) = e^{\lambda t} v(x)$ gives

$$\gamma v'' = \lambda v \qquad + \text{ homogeneous boundary conditions}$$

Check for nontrivial solutions:

- $\lambda > 0 \implies v(x) = Ae^{-\sqrt{\lambda/\gamma}x} + Be^{\sqrt{\lambda/\gamma}x} = \tilde{A}\cosh(\sqrt{\lambda/\gamma}x) + \tilde{B}\sinh(\sqrt{\lambda/\gamma}x)$
- $\lambda = 0 \implies v(x) = A + Bx$
- $\lambda < 0 \implies v(x) = A\cos(\sqrt{-\lambda/\gamma}x) + B\sin(\sqrt{-\lambda/\gamma}x)$

The superposition of all nontrivial solutions gives:

$$u(t,x) = \sum_{k=1}^{\infty} c_k e^{\lambda_k t} v_k(x) \qquad c_k = \frac{\int_a^b f(x)\overline{v_k(x)}\,\mathrm{d}x}{\int_a^b |v_k(x)|^2\,\mathrm{d}x}$$

Proposition *Green's function*

Consider for fixed $\sigma \in \mathbb{R}$ the boundary value problem

$$\begin{cases} \gamma v''(x) - \sigma v(x) = f(x) \\ \text{homogeneous boundary conditions at } x = a, b \end{cases}$$

Assume that the homogeneous boundary value problem only has the trivial solution:

$$f = 0 \implies v = 0$$

Then the boundary value problem has a **Green's function** $G : [a,b] \times [a,b] \to \mathbb{R}$ such that

$$v(x) = \int_a^b G(x;\xi) f(\xi)\,\mathrm{d}\xi$$

$Tf(x) = v(x)$ is a Fredholm operator, therefore if G is continuous, then T is compact.

Theorem *Spectral theorem for compact operators*

If T is a compact operator on a Banach space (for example L^2), then

1. For every $\varepsilon > 0$, the number of eigenvalues λ of T with $|\lambda| > \varepsilon$ is finite.
2. If $\lambda \neq 0$ is an eigenvalue of T, then $\dim\ker(T-\lambda) < \infty$

4.2 Boundary conditions on the wave equation

Recap

Wave equation: $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad u(0,x) = f(x) \qquad \frac{\partial u}{\partial t}(0,x) = g(x)$$

d'Alembert's formula: $$u(t,x) = \frac{f(x-ct)+f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,\mathrm{d}z$$

Proposition *Boundary conditions*

We apply the following Dirichlet boundary conditions to the wave equation:

$$u(t,0) = u(t,\ell) = 0$$

This gives the following solutions:

$$u(t,x) = \sum_{k=1}^{\infty}\left[a_k \cos\left(\frac{k\pi ct}{\ell}\right) + b_k \sin\left(\frac{k\pi ct}{\ell}\right)\right]\sin\left(\frac{k\pi x}{\ell}\right)$$

$$a_k = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\left(\frac{k\pi x}{\ell}\right)\mathrm{d}x \qquad b_k = \frac{2}{k\pi c}\int_0^{\ell} g(x)\sin\left(\frac{k\pi x}{\ell}\right)\mathrm{d}x$$

Periodic extension of d'Alembert's formula

Consider f and g from d'Alembert's formula. We have the following **odd periodic extension**:

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } 0 < x < \ell \\ -f(x) & \text{if } -\ell < x < 0 \\ 0 & \text{if } x \in \{-\ell, 0, \ell\} \end{cases} \qquad \tilde{f}(x+2\ell) = \tilde{f}(x) \text{ for all } x \in \mathbb{R}$$

We construct $\tilde{g}$ in the exact same way as $\tilde{f}$.

If we replace f and g with $\tilde{f}$ and $\tilde{g}$, then d'Alembert's formula satisfies the boundary conditions.
$\tilde{f}$ and $\tilde{g}$ have the same Fourier expansions as f and g.
Note: in some cases the boundary conditions and initial conditions are incompatible.

4.3 Planar Laplace equations

Definition *Laplace operator*

The 2-dimensional **Laplace operator** is given by

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

A function $u : \mathbb{R}^2 \to \mathbb{R}^2$ is called **harmonic** if $\Delta u = 0$.
Assuming $\Omega \subset \mathbb{R}^2$ is connected, open and bounded, we have the following boundary value problem:

$$\Delta u = 0 \text{ on } \Omega \qquad u = h \text{ on } \partial\Omega$$

Note that we can also replace the Dirichlet boundary condition with a different one.

Laplace equation on a rectangle

Consider the following problem:

$$\Delta u = 0 \text{ on } \Omega = (0,a) \times (0,b)$$

$$u(x,0) = f(x) \qquad u(x,b) = g(x) \qquad u(0,y) = h(y) \qquad u(a,y) = k(y)$$

Without loss of generality, we assume $g = h = k = 0$. We then have the following solution:

$$u(x,y) = \sum_{k=1}^{\infty} c_k \sin(\omega_k x) \sinh(\omega_k (b-y)) \qquad c_k = \frac{2}{a \sinh(\omega_k b)} \int_0^a f(x) \sin(\omega_k x) \, \mathrm{d}x \qquad \omega_k = \frac{k\pi}{a}$$

4.3.1 Laplace equation on a disk

Laplace equation on a disk

Consider the following problem:

$$\Delta u = 0 \text{ on } x^2 + y^2 < 1 \qquad u(x,y) = h(x,y) \text{ on } x^2 + y^2 = 1$$

We replace every occurrence of x and y by $r\cos\theta$ and $r\sin\theta$ respectively. This gives the following polar equation:

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial^2 \theta} = 0 \qquad u(1,\theta) = h(\theta)$$

Superposition of solutions <u>without singularities</u> at $r = 0$ gives:

$$u(r,\theta) = \frac{1}{\pi} \int_{-\pi}^{\pi} h(\phi) \left[\frac{1}{2} + \sum_{k=1}^{\infty} r^k \cos(k(\theta - \phi)) \right] \mathrm{d}\phi$$

Theorem *Poisson's formula*

For the Laplace equation on a disk, we have the following solutions:

$$u(r,\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - \phi)} h(\phi) \, \mathrm{d}\phi$$

In particular,

$$u(0,\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(\phi) \, \mathrm{d}\phi = \text{average value of } h$$

4.3.2 Average and maximum principle

Theorem *Average principle*

If u is harmonic inside the disk

$$D = \{(x,y) \in \mathbb{R}^2 : (x-a)^2 + (y-b)^2 \leq \rho^2\}$$

then we have

$$u(a,b) = \frac{1}{2\pi\rho} \oint_{\partial D} u \, \mathrm{d}s = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(a + \rho\cos\theta, b + \rho\sin\theta) \, \mathrm{d}\theta$$

Theorem *Maximum principle*

Assume that

- Ω is bounded, open and connected
- u is harmonic on Ω and continuous on $\partial\Omega$
- for all $(x,y) \in \Omega$ we have $u(x,y) \leq M$

If $u(x_0,y_0) = M$ for some $(x_0,y_0) \in \Omega$, then u is constant on Ω, and u attains its maximum value on $\partial\Omega$.

Corollary

Poisson's equation with Dirichlet conditions

$$-\Delta u = f \text{ on } \Omega \qquad u = h \text{ on } \partial\Omega$$

has a unique solution.

4.4 Classification of PDEs

Theorem

The solutions of

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

trace a curve whose type is determined by the **discriminant**

$$\Delta = B^2 - 4AC$$

The following classification applies:

- $\Delta > 0$: hyperbolic
- $\Delta = 0$: parabolic
- $\Delta < 0$: elliptic

Definition *Classification of PDEs*

For the linear, 2nd-order PDE

$$Au_{tt} + Bu_{tx} + Cu_{xx} + Du_t + Eu_x + F = 0$$

we define the **discriminant**

$$\Delta(t, x) = B^2 - 4AC$$

At a point (t, x), the PDE is called

- a **hyperbolic PDE** if $\Delta(t, x) > 0$
- a **parabolic PDE** if $\Delta(t, x) = 0$
- a **elliptic PDE** if $\Delta(t, x) < 0$
- a **singular PDE** if $A = B = C = 0$

5 Generalized functions

5.1 Dirac delta "function"

Lemma

Define:

$$r_n(x) = \begin{cases} n & \text{if } |x| \leq \frac{1}{2n} \\ 0 & \text{otherwise} \end{cases} \qquad f_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2x^2}$$

If u is continuous and $a < 0 < b$ then

$$\lim_{n\to\infty} \int_a^b r_n(x)u(x)\,\mathrm{d}x = u(0)$$

If u is continuous and bounded, then for all $\xi \in \mathbb{R}$ we have

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x-\xi)u(x)\,\mathrm{d}x = u(\xi)$$

Definition *Dirac delta function*

The **Dirac delta function** is defined by

$$\int_a^b \delta_\xi(x)u(x)\,\mathrm{d}x = u(\xi)$$

whenever u is continuous and $a < \xi < b$
Note: this is not actually a function, but it is a linear functional $u \mapsto u(\xi)$ on a suitable space.

Lemma

Any continuous function f is a "superposition" of delta functions:

$$f(x) = \int_{-\infty}^{\infty} f(\xi)\delta(x-\xi)d\xi$$

5.1.1 Generalized derivatives

Definition *Unit step function*

$$\sigma(x) = \int_{-1}^{x} \delta_0(t)\,\mathrm{d}t = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases} \qquad \sigma(0) = \frac{1}{x}\left(\lim_{x\to 0^-}\sigma(x) + \lim_{x\to 0^+}\sigma(x)\right) = \frac{1}{2} \qquad \frac{\mathrm{d}\sigma}{\mathrm{d}x} = \delta_0$$

Definition *Ramp function*

$$\rho(x) = \int_{-1}^{x} \sigma(t)\,\mathrm{d}t = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases} \qquad \frac{\mathrm{d}\rho}{\mathrm{d}x} = \sigma$$

Derivatives of the absolute value

$$f(x) = |x| \implies f'(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases} \implies f''(x) = 2\delta_0$$

General rule for generalized derivatives

If f has a discontinuity at $x = \xi$ such that

$$r := f(\xi^+) - f(\xi^-) \neq 0$$

then include the following term in the expression for f':

$$r\delta(x-\xi)$$

5.1.2 Fourier series of the delta function

Fourier coefficients of the delta function

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \delta(x)\cos(kx)\,\mathrm{d}x = \frac{1}{\pi}\cos(0) = \frac{1}{\pi} \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \delta(x)\sin(kx)\,\mathrm{d}x = \frac{1}{\pi}\sin(0) = 0$$

Definition *Dirac comb*

The **Dirac comb** is the 2π-periodic extension of δ:

$$\tilde{\delta}(x) = \sum_{k\in\mathbb{Z}} \delta(x - 2k\pi) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}\cos(kx) = \frac{1}{2\pi}\left(1 + 2\sum_{k=1}^{\infty}\cos(kx)\right)$$

5.2 Green's functions

General solution of a 2nd-order ODE

Consider the following ODE:

$$p(x)u'' + q(x)u' + r(x)u = f(x)$$

The general form of the solution is

$$u(x) = c_1 u_1(x) + c_2 u_2(x) + u_p(x)$$

where

- u_1, u_2 are linearly independent solutions of the homogeneous solution
- u_p is a particular solution of the inhomogeneous equation
- c_1, c_2 are constants

Variation of parameters

For a particular solution, try the ansatz $u_p = c_1 u_1 + c_2 u_2$, where c_1, c_2 depend on x.
By the product rule, we have

$$u_p' = c_1 u_1' + c_2 u_2' + c_1' u_1 + c_2' u_2$$

We then take a second ansatz: $c_1' u_1 + c_2' u_2 = 0$. Then we have:

$$u_p' = c_1 u_1' + c_2 u_2' \qquad u_p'' = c_1 u_1'' + c_2 u_2'' + c_1' u_1' + c_2' u_2'$$

Since u_1 and c_2 are solutions of the homogeneous equation, we get the following expression for f

$$f = pu_p'' + qu_p' + ru_p = p(c_1' u_1' + c_2' u_2') \implies c_1' u_1' + c_2' u_2' = \frac{f}{p} \implies \begin{bmatrix} u_1 & u_2 \\ u_1' & u_2' \end{bmatrix} \begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f/p \end{bmatrix}$$

Define the **Wronskian determinant** $w = u_1 u_2' - u_1' u_2$. Then we have

$$\begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \frac{1}{W} \begin{bmatrix} u_2' & -u_2 \\ -u_1' & u_1 \end{bmatrix} \begin{bmatrix} 0 \\ f/p \end{bmatrix} = \begin{bmatrix} -u_2/Wp \\ u_1 f/Wp \end{bmatrix}$$

c_1 and c_2 can then be found by integration.

Proposition

A particular solution to

$$p(x)u'' + q(x)u' + r(x)u = f(x)$$

is given by

$$u_p(x) = -u_1(x) \int \frac{u_2(x) f(x)}{W(x)p(x)} \, \mathrm{d}x + u_2(x) \int \frac{u_1(x) f(x)}{W(x)p(x)} \, \mathrm{d}x$$

where u_1, u_2 are solutions to the homogeneous case.

Proposition *Green's function for $u''(x) = f(x)$*

Consider the boundary value problem:

$$u''(x) = f(x) \qquad u(0) = u(1) = 0$$

This has the following solution:

$$\int_0^1 G(x,\xi) f(\xi) \, \mathrm{d}\xi \qquad G(x,\xi) = \begin{cases} (x-1)\xi & \text{if } \xi \geq x \\ (\xi - 1)x & \text{if } \xi \geq x \end{cases}$$

Properties of Green's function

Consider the following Green's function and its derivatives:

$$G(x,\xi) = \begin{cases} (x-1)\xi & \text{if } \xi \geq x \\ (\xi-1)x & \text{if } \xi \geq x \end{cases} \qquad \frac{\partial}{\partial x} G(x,\xi) = \begin{cases} \xi & \text{if } \xi < x \\ \xi - 1 & \text{if } \xi > x \end{cases} \qquad \frac{\partial^2}{\partial x^2} G(x,\xi) = \delta(x-\xi)$$

This Green's function satisfies the following properties:

- G solves the homogeneous equation $u''(x) = 0$ when $x \neq \xi$.
- G solves the homogeneous boundary conditions $u(0) = u(1) = 0$.
- G is continuous in (x, ξ).
- $\frac{\partial G}{\partial x}$ has a jump discontinuity at $x = \xi$.

Note: this Green's function can easily be derived by integrating its second derivative twice.

5.2.1 1-dimensional boundary value problems

Green's function for a 1-dimensional boundary value problem

Define the operator

$$L[u] = pu'' + qu' + ru$$

Goal: find a **Green's function** $G(x, \xi)$ such that

$$\begin{cases} L[u] = f \\ u(a) = u(b) = 0 \end{cases} \implies u(x) = \int_a^b G(x,\xi) f(\xi) \, \mathrm{d}\xi$$

We first find linearly independent solutions u_1, u_2 such that

$$L[u_1] = L[u_2] = 0 \qquad u_1(a) = u_2(b) = 0$$

Then we have the following ansatz:

$$G(x,\xi) = \begin{cases} c_1 u_1(x) & \text{if } x \leq \xi \\ c_2 u_2(x) & \text{if } x \geq \xi \end{cases}$$

We find c_1 and c_2 by requiring

- G is continuous at $x = \xi$
- $\dfrac{\partial G}{\partial x}$ has a jump discontinuity of magnitude $\dfrac{1}{p(\xi)}$ at $x = \xi$

5.2.2 Line integrals

Definition *Line integral*

Assume $t \mapsto (x(t), y(t))$ with $t \in [a, b]$ parametrizes a curve C.
The **line integral** of a scalar function f is

$$\int_C f \, \mathrm{d}s := \int_a^b f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2} \, \mathrm{d}t$$

The **line integral** of a vector field is

$$\int_C P \, \mathrm{d}x + Q \, \mathrm{d}y := \int_a^b P(x(t), y(t)) x'(t) + Q(x(t), y(t)) y'(t) \, \mathrm{d}t$$

Theorem *Green's theorem*

$$\int_{\partial\Omega} P \, \mathrm{d}x + Q \, \mathrm{d}y = \iint_\Omega \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, \mathrm{d}x \, \mathrm{d}y$$

Note: the parametrization of $\partial\Omega$ must satisfy the left-hand rule.

Proposition *Green's identities*

Let $\boldsymbol{n}$ denote the outward unit normal vector along $\partial\Omega$.

$$\textbf{Green's first identity}: \iint_\Omega u\Delta v + \nabla u \cdot \nabla v \,\mathrm{d}x\,\mathrm{d}y = \int_{\partial\Omega} u\frac{\partial v}{\partial \boldsymbol{n}}\,\mathrm{d}s$$

$$\textbf{Green's second identity}: \iint_\Omega u\Delta v - v\Delta u \,\mathrm{d}x\,\mathrm{d}y = \int_{\partial\Omega} u\frac{\partial v}{\partial \boldsymbol{n}} - v\frac{\partial u}{\partial \boldsymbol{n}}\,\mathrm{d}s$$

5.2.3 2-dimensional boundary value problems

Laplace operator (recap)

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \qquad f \text{ harmonic} \iff \Delta f = 0$$

Definition G_0

$$G_0(x,y,\xi,\eta) = -\frac{1}{2\pi}\log\|(x,y)-(\xi,\eta)\| = -\frac{1}{4\pi}\log[(x-\xi)^2+(y-\eta)^2]$$

From now on, consider (x,y) fixed and (ξ,η) variable.

Lemma

1. G_0 is harmonic on $\mathbb{R}^2 \setminus \{(x,y)\}$
2. If C_r is a circle of radius r around (x,y), then

$$G_0(x,y,\xi,\eta) = -\frac{1}{2\pi}\log r \qquad \frac{\partial G_0}{\partial \boldsymbol{n}}(x,y,\xi,\eta) = -\frac{1}{2\pi r} \qquad \text{for all } (\xi,\eta)\in C_r$$

Theorem *Representation formula*

Let $\boldsymbol{n}$ denote the outward unit normal vector along $\partial\Omega$. If u is harmonic in Ω, then for $(x,y)\in\Omega$ we have

$$u(x,y) = \int_{\partial\Omega} G_0\frac{\partial u}{\partial \boldsymbol{n}} - u\frac{\partial G_0}{\partial \boldsymbol{n}}$$

Definition G

$$G = G_0 + z \qquad \Delta z = 0 \text{ on } \Omega \qquad z = -G_0 \text{ on } \partial\Omega$$

Corollary

If u is a solution of

$$\Delta u = 0 \text{ on } \Omega \qquad u = h \text{ on } \partial\Omega$$

Then we have the representation formula

$$u(x,y) = -\int_{\partial\Omega} h\frac{\partial G}{\partial \boldsymbol{n}}\,\mathrm{d}s$$

Theorem

If u is a solution of

$$-\Delta u = f \text{ on } \Omega \qquad u = 0 \text{ on } \partial\Omega$$

Then we have the representation formula

$$u(x,y) = \iint_\Omega f(\xi,\eta)G(x,y,\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$$

Finding G using the method of images

To each point $(\xi, \eta) \in \Omega$ associate an **image point** $(\xi', \eta') \in \mathbb{R}^2 \setminus \overline{\Omega}$.
The following ansatz guarantees $\Delta z = 0$ on Ω:

$$z(x, y, \xi, \eta) = \frac{a}{2\pi} \log \|(x - y) - (\xi', \eta')\| + \frac{b}{2\pi}$$

Then we determine a, b and (ξ', η') such that

$$z(x, y, \xi, \eta) = -G_0(x, y, \xi, \eta) \text{ for all } (x, y) \in \partial\Omega$$

(for geometric examples, see the slides of Lecture 09.)

6 Fourier transforms

6.1 Fourier transforms

Definition L^1

$$L^1(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{C} : \int_{-\infty}^{\infty} |f(x)| \, \mathrm{d}x < \infty \right\}$$

Theorem *Fourier integral representation*

If $f \in L^1(\mathbb{R})$ is piecewise C^1, then

$$\frac{f(x^+) + f(x^-)}{2} = \int_0^{\infty} [A(k) \cos(kx) + B(k) \sin(kx)] \, \mathrm{d}k$$

$$A(k) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(y) \cos(ky) \, \mathrm{d}y \qquad B(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(y) \sin(ky) \, \mathrm{d}y$$

Definition *Fourier transform*

The **Fourier transform** of $f \in L^1(\mathbb{R})$ is given by

$$\widehat{f}(k) = \mathcal{F}[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} \, \mathrm{d}x$$

The **inverse Fourier transform** is given by

$$\mathcal{F}^{-1}[\widehat{f}(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(k) e^{ikx} \, \mathrm{d}k$$

Theorem

If $f \in L^1$ is piecewise C^1, then

$$\mathcal{F}^{-1}[\mathcal{F}[f(x)]] = \frac{f(x^+) + f(x^-)}{2}$$

6.2 Properties of Fourier transforms

6.2.1 Algebraic properties

Lemma

The Fourier transform is linear.

Lemma

If f is real and even, then $\widehat{f}$ is real and even.
If f is real and odd, then $\widehat{f}$ is purely imaginary and odd.

Lemma *Fourier transform of translations*

$$g(x) = f(x-\xi) \implies \widehat{g}(k) = e^{-ik\xi}\widehat{f}(k) \qquad g(x) = e^{i\xi x}f(x) \implies \widehat{g}(k) = \widehat{f}(k-\xi)$$

Lemma *Fourier transform of dilations*

$$g(x) = f(cx) \implies \widehat{g}(k) = \frac{1}{|c|}\widehat{f}\left(\frac{k}{c}\right)$$

Lemma *Symmetry principle*

$$\mathcal{F}[f(x)] = \widehat{f}(k) \implies \mathcal{F}[\widehat{f}(x)] = f(-k)$$

6.2.2 Derivatives

Lemma

If $f \in L^1(\mathbb{R})$ is C^1 and $\lim_{|x|\to\infty} f(x) = 0$, then

$$\widehat{(f')}(k) = ik\widehat{f}(k)$$

Lemma

If $f, xf \in L^1(\mathbb{R})$, then

$$\mathcal{F}[xf(x)] = i(\widehat{f})'(k)$$

Note

These two properties can be used to solve ODEs and PDEs, namely by applying a Fourier transform to the entire equation. Examples can be found at the end of the Lecture 10 slides.

6.2.3 Some "illegal" examples

Fourier transform of constant functions

$$\mathcal{F}[1] = \sqrt{2\pi}\delta(k) \qquad \mathcal{F}[e^{-ix\xi}] = \sqrt{2\pi}\delta(k+\xi)$$

Sign function

$$\text{sign}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases} \qquad \mathcal{F}[\text{sign}(x)] = -i\sqrt{\frac{2}{\pi}}\frac{1}{k}$$

Step function

$$\sigma(x) = \begin{cases} 1 & \text{if } x > 0 \\ \frac{1}{2} & \text{if } x = 0 \\ 0 & \text{if } x < 0 \end{cases} \qquad \mathcal{F}[\sigma(x)] = \sqrt{\frac{\pi}{2}}\delta(k) - \frac{i}{\sqrt{2\pi}k}$$

Inverse Fourier transform of rational functions

$$\mathcal{F}^{-1}\left[\frac{1}{k-a}\right] = i\sqrt{\frac{\pi}{2}}e^{iax}\,\text{sign}(x) \quad \text{if } a \in \mathbb{R} \qquad \mathcal{F}^{-1}\left[\frac{1}{k-(a+bi)}\right] = \begin{cases} -i\sqrt{2\pi}e^{(-b+ai)x}(1-\sigma(x)) & \text{if } b < 0 \\ i\sqrt{2\pi}e^{(-b+ai)x}\sigma(x) & \text{if } b > 0 \end{cases}$$

6.3 Convolutions

Definition *Convolution product*

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-\xi) g(\xi)\,\mathrm{d}\xi$$

Theorem *Properties of convolutions*

For functions f, g and constants a, b we have

1. $f * g = g * f$
2. $f * (g * h) = (f * g) * h$
3. $f * (ag + bh) = a(f * g) + b(f * h)$
4. $f * 0 = 0$
5. $f * \delta = f$
6. $f = f * g \implies \widehat{h} = \sqrt{2\pi}\widehat{f} \cdot \widehat{g}$
7. $f = f \cdot g \implies \widehat{h} = \frac{1}{\sqrt{2\pi}}\widehat{f} * \widehat{g}$

Note: properties 6 and 7 are very useful for solving PDEs.

6.4 Table of Fourier transforms

Table of Fourier transforms

$f(x)$	$\hat{f}(k)$
1	$\sqrt{2\pi}\,\delta(k)$
$\delta(x)$	$\frac{1}{\sqrt{2\pi}}$
$\sigma(x)$	$\sqrt{\frac{\pi}{2}}\,\delta(k) - \frac{i}{\sqrt{2\pi}\,k}$
sign x	$-i\sqrt{\frac{2}{\pi}}\frac{1}{k}$
$\sigma(x+a) - \sigma(x-a)$	$\sqrt{\frac{2}{\pi}}\frac{\sin ak}{k}$

$f(x)$	$\hat{f}(k)$
$e^{-ax}\,\sigma(x)$	$\frac{1}{\sqrt{2\pi}\,(a+ik)}$
$e^{ax}\,(1-\sigma(x))$	$\frac{1}{\sqrt{2\pi}\,(a-ik)}$
$e^{-a\lvert x\rvert}$	$\sqrt{\frac{2}{\pi}}\frac{a}{k^2+a^2}$
e^{-ax^2}	$\frac{e^{-k^2/(4a)}}{\sqrt{2a}}$
$\tan^{-1} x$	$\frac{\pi^{3/2}}{\sqrt{2}}\,\delta(k) - i\sqrt{\frac{\pi}{2}}\frac{e^{-\lvert k\rvert}}{k}$

$f(x)$	$\hat{f}(k)$
$f(cx+d)$	$\frac{e^{ikd/c}}{\lvert c\rvert}\hat{f}\left(\frac{k}{c}\right)$
$\hat{f}(x)$	$f(-k)$
$\hat{f}(-x)$	$f(k)$
$f'(x)$	$ik\,\hat{f}(k)$
$xf(x)$	$i\,\hat{f}\,'(k)$
$f * g(x)$	$\sqrt{2\pi}\,\hat{f}(k)\,\hat{g}(k)$

6.5 Fundamental solution of the heat equation

Nonhomogeneous initial value problem

Consider the following problem:

$$\frac{\partial u}{\partial t} = \gamma\frac{\partial^2 u}{\partial x^2} + h(t,x) \qquad t > 0 \qquad -\infty < x < \infty \qquad u(0,x) = f(x)$$

We can split it into two problems as follows:

$$\frac{\partial v}{\partial t} = \gamma\frac{\partial^2 v}{\partial x^2} \quad v(0,x) = f(x) \qquad \frac{\partial w}{\partial t} = \gamma\frac{\partial^2 w}{\partial x^2} + h(t,x) \quad w(0,x) = 0$$

and then obtain the solutions by doing Fourier transforms.

Unforced equation

In the case $f(x) = \delta(x-\xi)$ we have the following **fundamental solution** for the v problem:

$$F(t,x;\xi) = \frac{1}{2\sqrt{\pi\gamma t}} e^{-(x-\xi)^2/4\gamma t}$$

Let $F_0(t,x) = F(t,x;0)$. For general f, we have the following solution

$$v(t,x) = (F_0 * f)(x,t)$$

Forced equation

In the case $h(t,x) = \delta(t-\tau)\delta(x-\xi)$ we have the following **fundamental solution** for the w problem:

$$G(t,x;\tau,\xi) = \frac{\sigma(t-\tau)}{2\sqrt{\pi\gamma(t-\tau)}} e^{-\frac{(x-\xi)^2}{4\gamma(t-\tau)}} = \sigma(t-\tau)F(t-\tau,x;\xi)$$

For general $h(x,t)$, we have the following solution

$$w(t,x) = \int_0^t \int_{-\infty}^{\infty} \frac{h(\tau,\xi)}{2\sqrt{\pi\gamma(t-\tau)}} e^{-\frac{(x-\xi)^2}{4\gamma(t-\tau)}} \,\mathrm{d}\xi \,\mathrm{d}\tau$$

General solution of the forced heat equation

$$u(t,x) = \frac{1}{2\sqrt{\gamma\pi t}} \int_{-\infty}^{\infty} e^{-(x-\xi)^2/4\gamma t} f(\xi) \,\mathrm{d}\xi + \int_0^t \int_{-\infty}^{\infty} \frac{h(\tau,\xi)}{2\sqrt{\gamma\pi(t-\tau)}} e^{-(x-\xi)^2/4\gamma(t-\tau)} \,\mathrm{d}\xi \,\mathrm{d}\tau$$

6.5.1 Maximum principle

Definition *Rectangle with (partial) boundary*

$$R = \{(t,x) \in \mathbb{R}^2 : 0 < t < c \text{ and } a < x < b\}$$

$$R_1 = \{(t,x) \in \mathbb{R}^2 : t = 0 \text{ and } a < x < b\} \qquad R_2 = \{(t,x) \in \mathbb{R}^2 : 0 < t < c \text{ and } x = a\}$$

$$R_3 = \{(t,x) \in \mathbb{R}^2 : 0 < t < c \text{ and } x = b\} \qquad B = R_1 \cup R_2 \cup R_3$$

Theorem *Maximum principle*

Assume $\gamma > 0$, $h(t,x) \le 0$ for all $(t,x) \in R$, and u satisfies the following PDE on R:

$$\frac{\partial u}{\partial t} = \gamma \frac{\partial^2 u}{\partial x^2} + h(t,x)$$

Then the maximum of u on $\overline{R}$ is attained on the set B.

Corollary

Assume that u satisfies

$$\frac{\partial u}{\partial t} = \gamma \frac{\partial^2 u}{\partial x^2}$$

and define

$$m = \min\{u(t,x) : (t,x) \in B\} \qquad M = \max\{u(t,x) : (t,x) \in B\}$$

Then for all $(t,x) \in R$, we have

$$m \le u(t,x) \le M$$

Theorem

The problem

$$\frac{\partial u}{\partial t} = \gamma \frac{\partial^2 u}{\partial x^2} + h(t,x)$$

with initial condition and Dirichlet boundary conditions at $x = a, b$ has at most 1 solution.

7 Nonlinear PDEs

7.1 Nonlinear transport equation

Note

Nonlinear PDEs do not satisfy the superposition principle.

Nonlinear transport equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} \qquad u(0,x) = f(x)$$

Definition *Characteristic curve (nonlinear)*

Characteristic curves of the nonlinear transport equation are graphs of solutions of

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u(t, x(t))$$

Proposition

A solution u of the nonlinear transport equation is constant along characteristic curves.

Corollary *Implicit expression for u*

$$u(t, f(y)t + y) = f(y)$$

7.2 Nonlinear diffusion

Burgers' equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \gamma\frac{\partial^2 u}{\partial x^2} \qquad \gamma > 0$$

Solution of Burgers' equation

The ansatz $u(t,x) = v(\xi) = v(x - ct)$ where v is bounded leads to the following solution:

$$u(t,x) = \frac{ae^{(b-a)(x-ct-\delta)/(2\gamma)} + b}{e^{(b-a)(x-ct-\delta)/(2\gamma)} + 1}$$

Proposition *Hopf-Cole transformation*

All solutions to Burgers' equation can be obtained via the transformation

$$u(t,x) = \frac{\partial}{\partial x}[-2\gamma \log v(t,x)] = -2\gamma\frac{v_x(t,x)}{v(t,x)}$$

where v is a positive solution to the heat equation.

Proposition *Solution of Burgers' equation with initial condition*

The solution of Burgers' equation with initial condition $u(0,x) = f(x)$ is given by

$$u(t,x) = -2\gamma\frac{\widetilde{v}_x(t,x)}{\widetilde{v}(t,x)} \qquad \tilde{v}(t,x) = \int_{-\infty}^{\infty} e^{-H(t,x;\xi)}\,\mathrm{d}\xi \qquad H(t,x;\xi) = \frac{(x-\xi)^2}{4\gamma t} + \frac{1}{2\gamma}\int_0^{\xi} f(\eta)\,\mathrm{d}\eta$$

7.3 Dispersion

Dispersive wave equation with boundary conditions

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} = 0 \qquad t > 0 \qquad -\pi < x < \pi \qquad u(0, x) = f(x)$$

$$u(t, -\pi) = u(t, \pi) \qquad u_x(t, -\pi) = u_x(t, \pi) \qquad u_{xx}(t, -\pi) = u_{xx}(t, \pi)$$

We have the following Fourier series solution:

$$u(t, x) = \sum_{k=-\infty}^{\infty} c_k(0) e^{i(kx + k^3 t)} \qquad c_k(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} \, \mathrm{d}x$$

Definition *Airy function of the first kind*

$$\mathrm{Ai}(x) = \frac{1}{\pi} \int_0^{\infty} \cos(kx + \frac{1}{3}k^3) \, \mathrm{d}k$$

Linearized Korteweg-de Vries equation

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} = 0 \qquad u(0, x) = f(x) = \delta_\xi(x)$$

By taking Fourier transforms we get a fundamental solution:

$$F(t, x; \xi) = \mathcal{F}^{-1}\left[\widehat{\delta_\xi}(k) e^{ik^3 t}\right] = \frac{1}{\sqrt[3]{3t}} \mathrm{Ai}\left(\frac{x - \xi}{\sqrt[3]{3t}}\right)$$

The solution for arbitrary $f(x)$ is

$$u(t, x) = (F_0 * f)(t, x) = \frac{1}{\sqrt[3]{3t}} \int_{-\infty}^{\infty} \mathrm{Ai}\left(\frac{x - \xi}{\sqrt[3]{3t}}\right) f(\xi) \, \mathrm{d}\xi$$

Korteweg-de Vries equation

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + u\frac{\partial u}{\partial x} = 0 \qquad t > 0 \qquad -\infty < x < \infty$$

As an ansatz, we take a traveling wave with some conditions on the asymptotic behavior.

$$u(t, x) = v(\xi) = v(x - ct) \qquad \lim_{\xi \to \pm\infty} v(\xi) = 0 \qquad \lim_{\xi \to \pm\infty} v'(\xi) = 0 \qquad \lim_{\xi \to \pm\infty} v''(\xi) = 0$$

This leads to the following ODE:

$$(v')^2 = v^2 \left(c - \frac{1}{3}v\right)$$

A solution is then given by

$$u(t, x) = 3c \,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{c}(x - ct) + \delta\right) \qquad \mathrm{sech}(y) = \frac{2}{e^{-y} + e^{y}}$$

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